UNIVERSITY OF MARYLAND

Technical Note BN-605 1969

POTENTIAL DISTRIBUTION SURROUNDING A PHOTO-EMITTING PLATE IN A DILUTE PLASMA

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Ralph L. Guernsey and Jerry H.M. Fu

University of Maryland

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THE INSTITUTE FOR FLUID DYNAMICS *and* APPLIED MATHEMATICS

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ABSTRACT

The potential distribution in the neighborhood of a photo-emitting plate immersed in a plasma is calculated. We find that two steady state potential distributions can exist, namely one in which the potential decreases from its plate value to zero monotonically and one in which it decreases from its plate value to a (negative) minimum and then increases slowly to zero. This latter "over shoot" potential appears to be the stable one. Such over shoot effects are expected to play an important role for sheats around satellites in the interplanetary plasma.

INTRODUCTION

When a satellite is illuminated by the ultra-violet radiation of the sun, photo-electrons are emitted. In the interplanetary medium, this photo-electric effect dominates the accretion of thermal electrons, so that a positive surface potential is established[1,2]. This positive potential attracts electrons and forms a steady state electron sheath.

In the present work we consider the potential in the neighborhood of the satellite. Two possibilities can arise. The electrostatic potential might decrease monotonically to zero, or it might over-shoot[3] (Fig.1.). The present work is an investigation of the conditions under which the potential distribution may take the form of Fig.1, and the stability of this solution relative to the monotonic solution.

The problem is approached using one dimensional model subject to the following restrictions:

(1) The satellite surface is idealized to a large (compared to the sheath dimension) metal plate which emits photo-electrons, which have a velocity distribution $f_\nu(v_o, 0)$ at the plate.

(2) The cold plasma-ions are drifting toward the plate with a velocity U_i, which is much smaller than the thermal velocity of the plasma-electrons. But due to the large ion-mass, the kinetic energy of an ion is much greater than any potential inside the sheath, so no ions are reflected.

(3) The temperature of the plasma-electrons is finite. The velocity distribution of the incoming plasma-electrons at the sheath edge is $f_e(v_\infty, \infty)$.

(4) The steady state is defined by the absence of any net current.

(5) An over-shoot electrostatic potential is assumed (Fig.1), but the surface potential, ϕ_o , and the minimum potential, ϕ_m , are left to be determined.

II. PARTICLE DYNAMICS

(a) Photo-electron Dynamics

In the system we are considering, it is well known that the total energy of every photo-electron is constant (the constant may differ from one electron to the other). The energy equation for a typical photo-electron is

$$\frac{m}{2} v^2 - e\phi(x) = \frac{m}{2} v_o^2 - e\phi_o \tag{1}$$

where

ϕ_o is the potential at $x = 0$ (the surface potential),

v_o is the speed of the photo-electron at $x = 0$,

$\phi(x)$ is the potential at x and

v is the speed of the photo-electron at position x .

According to the energy relation, eq. (1), the photo-electrons at $x = 0$ consist of two sets: 1.) those with velocity range

$$- \sqrt{\frac{2e}{m}\left(\phi_o - \phi_m\right)} < v_o < \sqrt{\frac{2e}{m}\left(\phi_o - \phi_m\right)} \tag{2}$$

are trapped in the region between $0 \leq x < x_m$ 2.) those with velocity range

$$\sqrt{\frac{2e}{m}\left(\phi_o - \phi_m\right)} < v_o < \infty \tag{3}$$

can escape. But from eq. (1), eqs (2) and (3) are equivalent to

$$\left.\begin{aligned} - \sqrt{\frac{2e}{m}\left(\phi - \phi_m\right)} < v < \sqrt{\frac{2e}{m}\left(\phi - \phi_m\right)} \\ 0 \leq x < x_m \end{aligned}\right\} \tag{2-1}$$

and

$$\left.\begin{aligned} \sqrt{\frac{2e}{m}\left(\phi - \phi_m\right)} < v < \infty \\ 0 \leq x \leq \infty \end{aligned}\right\} \tag{3-1}$$

hence the number density of photo-electrons at position x is

$$N_\nu(x) = S\left(x_m - x\right) \int_{-\sqrt{\frac{2e}{m}(\phi - \phi_m)}}^{\sqrt{\frac{2e}{m}(\phi - \phi_m)}} dv \quad f_\nu(v,x) + \int_{\sqrt{\frac{2e}{m}(\phi - \phi_m)}}^{\infty} dv \; f_\nu(v,x) \tag{4}$$

where $S(a)$ is a step function, and $f_\nu(v,x)$ is the velocity distribution at x. Through the energy relation, $f_\nu(v,x)$ is related to its boundary distribution, $f_\nu(v_o,0)$, by

$$f_\nu(v,x) = S\left(x_m - x\right) \int_{\sqrt{\frac{2e}{m}(\phi_o-\phi)}}^{\sqrt{\frac{2e}{m}(\phi_o-\phi_m)}} dv_o\ f_\nu\left(v_o,0\right)\ \delta\left[v_o - \sqrt{v^2 + \frac{2e}{m}\left(\phi_o - \phi\right)}\right]$$

$$+ \ S(v) \int_{\sqrt{\frac{2e}{m}(\phi_o-\phi_m)}}^{\infty} dv_o\ f_\nu\left(v_o,0\right)\ \delta\left[v_o - \sqrt{v^2 + \frac{2e}{m}\left(\phi_o - \phi\right)}\right] \tag{5}$$

where $\delta(b)$ is the usual Delta-function.

Substituting eq. (5) into eq. (4), changing the order of integration and using the properties of δ-function, we find

$$N_\nu(x) = 2\ S\left(x_m - x\right) \int_{\sqrt{\frac{2e}{m}(\phi_o-\phi)}}^{\sqrt{\frac{2e}{m}(\phi_o-\phi_m)}} dv_o\ f_\nu\left(v_o,0\right) \frac{v_o}{\sqrt{v_o^2 - \frac{2e}{m}(\phi_o-\phi)}} +$$

$$+ \int_{\sqrt{\frac{2e}{m}(\phi_o-\phi_m)}}^{\infty} dv_o\ f_\nu\left(v_o,0\right) \frac{v_o}{\sqrt{v_o^2 - \frac{2e}{m}(\phi_o-\phi)}} \tag{6}$$

(b) Plasma-electron Dynamics

The energy equation for a typical plasma-electron is

$$\frac{m}{2} v_\infty^2 = \frac{m}{2} v^2 - e\phi(x) \tag{7}$$

where v_∞ is the speed of the plasma-electron at ∞ , where the potential is assumed to be zero. Through analysis, similar to that used for the photo-electrons dynamics, the distribution function and the number density of the plasma-electrons at position $x(x < \infty)$ are

$$f_e(v,x) = S(x - x_m) \int_{-\sqrt{-\frac{2e}{m}\phi_m}}^{-\sqrt{-\frac{2e}{m}\phi}} dv_\infty\, f_e(v_\infty,\infty)\, \delta\left[v_\infty + \sqrt{v^2 - \frac{2e}{m}\phi}\right]$$

$$+ \quad S(-v) \int_{-\infty}^{-\sqrt{-\frac{2e}{m}\phi_m}} dv_\infty\, f_e(v_\infty,\infty)\, \delta\left[v_\infty + \sqrt{v^2 - \frac{2e}{m}\phi}\right] \tag{8}$$

and

$$N_e(x) = 2S(x - x_m) \int_{\sqrt{-\frac{2e}{m}\phi}}^{\sqrt{-\frac{2e}{m}\phi_m}} dv_\infty \frac{v_\infty\, f_e(-v_\infty,\infty)}{\sqrt{v_\infty^2 + \frac{2e}{m}\phi}} \quad +$$

$$+ \int_{\sqrt{-\frac{2e}{m}\phi_m}}^{\infty} dv_\infty \frac{v_\infty\, f_e(-v_\infty,\infty)}{\sqrt{v_\infty^2 + \frac{2e}{m}\phi}} \quad . \tag{9}$$

(c) Plasma-ion Dynamics

In general, the energy equation for an ion is

$$\frac{M_i}{2} v_\infty^2 = \frac{M_i}{2} v^2 + e\phi(x) \tag{10}$$

The cold ion assumption implies

$$N_i(x) = \frac{n_i}{\sqrt{1 - \frac{2e\phi(x)}{M_i U_i^2}}} \tag{11}$$

where n_i is the number density of plasma-ions at ∞. Since the kinetic energy of an ion is much greater than the sheath potential [see assumption (2) above], the density of the ions may be taken as constant for all x approximately.

III. POISSON'S EQUATION

By collecting the appropriate terms obtained in the last section, the Poisson's equation can be written as

$$\frac{d^2\phi(x)}{dx^2} = -4\pi\rho(x) = 4\pi e \begin{cases} P_1[\phi(x)] & x \leqslant x_m \\ \\ P_2[\phi(x)] & x \geqslant x_m \end{cases} \tag{12}$$

where

$$P_1(\phi) \equiv - n_i + \int_{\sqrt{\frac{2e}{m}(\phi_o - \phi_m)}}^{\infty} dv_o \frac{v_o \, f_\nu(v_o,0)}{\sqrt{v_o^2 - \frac{2e}{m}(\phi_o - \phi)}} + \int_{\sqrt{-\frac{2e}{m}\phi_m}}^{\infty} dv_\infty \frac{v_\infty f_e(-v_\infty,\infty)}{\sqrt{v_\infty^2 + \frac{2e}{m}\phi}} +$$

$$+ \; 2 \int_{\sqrt{\frac{2e}{m}(\phi_o - \phi)}}^{\sqrt{\frac{2e}{m}(\phi_o - \phi_m)}} dv_o \frac{v_o \, f_\nu(v_o,0)}{\sqrt{v_o^2 - \frac{2e}{m}(\phi_o - \phi)}} \, , \tag{13}$$

and

$$P_2(\phi) \equiv - n_i + \int_{\sqrt{\frac{2e}{m}(\phi_o - \phi_m)}}^{\infty} dv_o \frac{v_o \, f_\nu(v_o,0)}{\sqrt{v_o^2 - \frac{2e}{m}(\phi_o - \phi)}} + \int_{\sqrt{-\frac{2e}{m}\phi_m}}^{\infty} dv_\infty \frac{v_\infty f_e(-v_\infty,\infty)}{\sqrt{v_\infty^2 + \frac{2e}{m}\phi}} +$$

$$+ \; 2 \int_{\sqrt{-\frac{2e}{m}\phi}}^{\sqrt{-\frac{2e}{m}\phi_m}} dv_\infty \frac{v_\infty f_e(-v_\infty,\infty)}{\sqrt{v_\infty^2 + \frac{2e}{m}\phi}} \, , \tag{14}$$

The first and the second integrals in $P_1(\phi)$ and $P_2(\phi)$ are due to free photo-electrons and free plasma-electrons; while the last integral in $P_1(\phi)$ is due to "trapped" photo-electrons and the last one in $P_2(\phi)$ is due to reflected plasma-electrons.

Now, we shall cast the Poisson's equation into the form of the equation of motion for a fictitious "particle" moving in a potential "well". This is done by differentiating by parts, and by applying the Leibniz' formula for differentiation.

$$\frac{d^2\phi}{dx^2} = \begin{cases} -\dfrac{\partial}{\partial\phi}\overline{V}_1(\phi) & x \leqslant x_m \\ \\ -\dfrac{\partial}{\partial\phi}\overline{V}_2(\phi) & x \geqslant x_m \end{cases} \tag{15}$$

where

$$\begin{aligned} \overline{V}_1(\phi) \equiv + 2\pi m \Bigg\{ & n_i \frac{2e}{m}\left(\phi - \phi_m\right) \\ & - 2\int_{\sqrt{-\frac{2e}{m}\phi_m}}^{\infty} dv'\, v' \left[\sqrt{v'^2 + \frac{2e}{m}\phi} - \sqrt{v'^2 + \frac{2e}{m}\phi_m}\right]\left[f_\nu\left(\sqrt{v'^2 + \frac{2e}{m}\phi_o}, 0\right)\right. \\ & \left. + f_e\left(-v', \infty\right)\right] - 4\int_{\sqrt{\frac{2e}{m}(\phi_o - \phi)}}^{\sqrt{\frac{2e}{m}(\phi_o - \phi_m)}} dv'\, v' \sqrt{v'^2 - \frac{2e}{m}(\phi_o - \phi)}\; f_\nu(v', 0) \Bigg\} \end{aligned} \tag{16}$$

and

$$\overline{V}_2(\phi) \equiv +2\pi m \Bigg\{ n_i \frac{2e}{m}\left(\phi - \phi_m\right)$$

$$- 2\int_{\sqrt{-\frac{2e}{m}\phi_m}}^{\infty} dv'\, v' \left[\sqrt{v'^2 + \frac{2e}{m}\phi} - \sqrt{v'^2 + \frac{2e}{m}\phi_m}\right]\left[f_\nu\left(\sqrt{v'^2 + \frac{2e}{m}\phi_o}, 0\right) + f_e\left(-v', \infty\right)\right] \tag{17}$$

$$- 4\int_{\sqrt{-\frac{2e}{m}\phi}}^{\sqrt{-\frac{2e}{m}\phi_m}} dv'\, v' \sqrt{v'^2 + \frac{2e}{m}\phi}\; f_e(-v', \infty)\Bigg\} \quad .$$

Here we have choosen the arbitrary constants implicit in defining $\overline{V}_1$, $\overline{V}_2$ from (15) by requiring that

$$\overline{V}_1(\phi_m) = \overline{V}_2(\phi_m) = 0 \quad ,$$

and have changed the variable of integration. The integration of eq. (15) from x_m to x , yields

$$\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 = \begin{cases} -\overline{V}_1(\phi) & x \leqslant x_m \\ -\overline{V}_2(\phi) & x \geqslant x_m \end{cases} \tag{18}$$

Eq. (18) can be reduced to quadrature

$$\int_{\phi_m}^{\phi} \frac{d\phi'}{\sqrt{-\overline{V}_1(\phi)}} = -\sqrt{2}\left(x - x_m\right) \quad x \leqslant x_m \tag{19}$$

$$\int_{\phi_m}^{\phi} \frac{d\phi'}{\sqrt{-\overline{V}_2(\phi)}} = \sqrt{2}\left(x - x_m\right) \quad x \geqslant x_m \tag{19-1}$$

Note that $\overline{V}_1(\phi)$, $\overline{V}_2(\phi)$ are both negative, this is due to the fact that ϕ is bounded in our problem so that the fictitious "particle" must move in a potential "well".

Now we examine the conditions which are imposed by our assumptions.

(1) Neutrality condition at ∞ [from eq. (14)]

$$n_i = \int_{\sqrt{-\frac{2e}{m}\phi_m}}^{\infty} dv' \left[f_\nu\left(\sqrt{v'^2 + \frac{2e}{m}\phi_o}, 0\right) + f_e(-v', \infty)\right] + 2\int_0^{\sqrt{-\frac{2e}{m}\phi_m}} dv' \; f_e(v', \infty) \; . \tag{20}$$

(2) No electric field at ∞ [from eqs. (18), (17), and (20)]

$$\int_{\sqrt{-\frac{2e}{m}\phi_m}}^{\infty} dv' \left(v' - \sqrt{v'^2 + \frac{2e}{m}\phi_m}\right)^2 \left[f_\nu\left(\sqrt{v'^2 + \frac{2e}{m}\phi_o}, 0\right) + f_e(-v', \infty)\right]$$
$$- 4\int_0^{\sqrt{-\frac{2e}{m}\phi_m}} dv' \left[\frac{1}{2}\left(-\frac{2e}{m}\phi_m\right) - v'^2\right] f_e(-v', \infty) = 0 \quad . \tag{21}$$

For $\phi_m = 0$, both terms vanish independently. However, for some distributions f_ν , f_e , we will see that there exists a $\phi_m < 0$ such that eq. (21) holds (i.e. over-shoot is possible).

(3) To fulfill the requirement that the potential is a minimum at x_m , it is necessary to have a deficiency of ions at x_m $\left(\text{i.e. } \left.\frac{d^2\phi}{dx^2}\right|_{x_m} > 0\right)$. From eqs. (12), (13) [or (14)], and (20), this condition can be written as

$$\int_{\sqrt{-\frac{2e}{m}\phi_m}}^{\infty} dv' \left[\frac{v'}{\sqrt{v'^2 + \frac{2e}{m}\phi_m}} - 1\right] \left[f_\nu\left(\sqrt{v'^2 + \frac{2e}{m}\phi_o}, 0\right) + f_e(-v',\infty)\right] > 2\int_0^{\sqrt{-\frac{2e}{m}\phi_m}} dv' \; f_e(-v',\infty) \tag{22}$$

If the photo-electrons are completely "trapped", and the plasma-electrons have a Maxwellian distribution at ∞ one can easily show, via (22), that the over-shoot is impossible.*

(4) Initially, the photo-electrons-flux is much greater than the plasma-electrons-flux which in turn is much greater than the plasma-ions-flux, i.e.

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* for this case eq. (22) becomes

$$1 + \text{erf}\left(\sqrt{-e\beta\phi_m}\right) < e^{e\beta\phi_m} \text{ , which cannot be satisfied for } \phi_m < 0 \text{ .}$$

$$\int_0^{\infty} dv_o v_o \, f_\nu(v_o,0) \gg \left| \int_{-\infty}^{0} dv_\infty v_\infty \, f_e(v_\infty,\infty) \right| \gg \left| - n_i U_i \right| \quad . \tag{23}$$

(5) For steady state, we insist on zero total current.

Since the "trapped" photo-electrons and the reflected plasma-electrons contribute no net current, this condition can be written as

$$\int_{\sqrt{\frac{2e}{m}(\phi_o-\phi_m)}}^{\infty} dv_o v_o f_\nu(v_o,0) + \int_{-\infty}^{-\sqrt{-\frac{2e}{m}\phi_m}} dv_\infty v_\infty \, f_e(v_\infty,\infty) + n_i U_i = 0 \tag{24}$$

where we have used the fact that the steady state current of every species is independent of position.

Equations (22) and (23) are conditions imposed on the distribution functions. Given a set of distribution functions which satisfy these conditions; then the surface potential, ϕ_o , and the over-shoot potential minimum, ϕ_m , can be determined via eqs. (20), (21), and (24).

IV. NON-MONOTONIC SOLUTION

In order to explore the possibility of steady-state potential distributions taking the form of Fig. 1, we now consider the special case* in which the distribution of emitted photo-electrons at $x = 0$ has the one-dimension Fermi shape (Fig.2), and the distribution of incoming plasma electrons at ∞ is Maxwellian, namely

$$f_\nu(v_o,0) = \frac{2n_\nu v_o}{u_o^2} S(u_o - v_o) \text{ , (Fig.2.)} \tag{25}$$

$$f_e(v_\infty,\infty) = n_e\sqrt{\frac{m\beta}{2\pi}}\exp\left(\frac{-\beta m v_\infty^2}{2}\right) \tag{26}$$

where the characteristic velocity of the photo-electrons, u_o, is assumed† to be greater than $\sqrt{\frac{2e}{m}(\phi_o-\phi_m)}$, and $S(a)$ is the usual step-function.

With these distributions, eqs. (23), (24), and (21) reduce to

$$\frac{3}{2}\left(\frac{n_i u_i}{n_\nu u_o}\right) << \frac{3}{\sqrt{8\pi}}\left(\frac{n_e}{n_\nu\sqrt{m\beta u_o^2}}\right) << 1 \quad , \tag{27}$$

$$\frac{3}{2}\left[1 - (1-\varepsilon^2)^{3/2}\right] + \frac{n_i u_i}{n_\nu u_o} = \frac{1}{\sqrt{2\pi}}\;\frac{n_e \exp(-z)}{n_\nu\sqrt{m\beta u_o^2}} \quad , \tag{28}$$

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* The results are expected to be qualitatively valid for any "one-parameter" shapes for $f_\nu(v_o,0), f_e(v_\infty,\infty)$, i.e. those characterized by single macroscopic velocities (in this case u_o and $(\beta m)^{-1/2}$).

† There can be no steady state for the opposite assumption.

and

$$G(\varepsilon,\delta) \quad + \quad \frac{n_e}{2\beta e} \quad F(z) \quad = \quad 0 \quad , \tag{29}$$

where

$$\begin{aligned} G(\varepsilon,\delta) \equiv \frac{n_\nu m u_o^2}{4e} \Bigg\{ & 2\left[\left(\varepsilon^2 + \delta\right)^{3/2} - \sqrt{1-\varepsilon^2}\,\delta^{3/2} - \varepsilon^3\right] \\ & + (1-\varepsilon^2-\delta)\left(\sqrt{\varepsilon^2+\delta} - \sqrt{1-\varepsilon^2}\,\sqrt{\delta}\right) - (1-\varepsilon^2)\varepsilon \\ & - (1-\varepsilon^2-\delta)^2\,\ell n\left(\frac{1+\sqrt{\varepsilon^2+\delta}}{\sqrt{1-\varepsilon^2}+\sqrt{\delta}}\right) + (1-\varepsilon^2)^2\,\ell n\left(\frac{1+\varepsilon}{\sqrt{1-\varepsilon^2}}\right) \\ & - 2\delta\left[\sqrt{\varepsilon^2+\delta} - \sqrt{\delta}\,\sqrt{1-\varepsilon^2} + (1-\varepsilon^2-\delta)\,\ell n\left(\frac{1+\sqrt{\varepsilon^2+\delta}}{\sqrt{1-\varepsilon^2+\delta}}\right)\right]\Bigg\} \end{aligned} \tag{30}$$

$$F(z) \equiv (1-z)\left[1+\mathrm{erf}(\sqrt{z})\right] - \left(1+2\sqrt{\frac{z}{\pi}}\right)\exp(-z) \tag{31}$$

with

$$\varepsilon \quad \equiv \quad \sqrt{1 - \frac{2e(\phi_o - \phi_m)}{m u_o^2}} \tag{32}$$

$$\delta \quad \equiv \quad - \quad \frac{-\,2e\phi_m}{m u_o^2} \tag{33}$$

$$z \quad \equiv \quad - \quad e\beta\phi_m \tag{34}$$

In view of the condition eq. (27), eq. (28) cannot be satisfied unless ε is small, which also implies $\delta < 1$. Thus one may expand the terms in G, to obtain*

$$G(\varepsilon,\delta) \cong \frac{n_\nu m u_o^2}{e}\left\{\frac{2}{3}\left[(\varepsilon^2+\delta)^{3/2}-\varepsilon^3\right]+\frac{1}{3}\delta^{3/2}-\delta\sqrt{\varepsilon^2+\delta}+O\left((\varepsilon^2+\delta)^2\right)\right\}$$

$$\delta \ll 1 , \quad \varepsilon \ll 1 \quad . \tag{35}$$

Based on the assumption that $u_i \ll (m\beta)^{-1} \ll u_o$, one may neglect the term proportional to u_i in eq. (28). Then eq. (28) reduces to

$$\varepsilon^2 \cong \frac{1}{\sqrt{2\pi}} \frac{n_e}{n_\nu\sqrt{m\beta u_o^2}} \exp(-z) \quad , \tag{36}$$

and eq. (29) reduces to

$$\frac{2}{3}\left[(\varepsilon^2+\delta)^{3/2}-\varepsilon^3\right]-\delta\sqrt{\delta+\varepsilon^2} = -\frac{\delta}{3}^{3/2} - \frac{n_e F(z)}{2n_\nu \beta m u_o^2} \tag{37}$$

It is clear that δ is related to z through eqs. (33) and (34). Therefore eqs. (36) and (37) are two algebraic equations for ε and z (i.e. for ϕ_o, ϕ_m). To simplify this set of equations, we use the relation between δ and z, and define

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* By expanding eq. (30) also for $\delta \gg \varepsilon^2$, one readily shows that eq. (35) gives the correct leading terms for all δ with an error no worse than $\varepsilon^{2/3}$ times leading term.

$$\varepsilon \equiv \sqrt{\frac{2z}{\beta m u_o^2}}\, y \tag{38}$$

and

$$\lambda \equiv \frac{n_e}{n_\nu} \sqrt{\beta m u_o^2} \tag{39}$$

Then eqs. (36) and (37) reduce to

$$\lambda = \sqrt{8\pi}\, z \exp(-z)\, y^2 \tag{40}$$

and

$$\sqrt{y^2+1}\left(y^2 - \frac{1}{2}\right) = y^3 - \frac{1}{2} - \frac{3}{4}\,\frac{\lambda F(z)}{(2z)^{3/2}} \tag{41}$$

Although λ is a known parameter (in principle) and y , z are to be determined, it is convenient to temporarily eliminate λ from eqs. (40) and (41) to obtain

$$\sqrt{y^2+1}\left(y^2 - \frac{1}{2}\right) = y^3 - \frac{1}{2} + \frac{3}{4}\, y^2\, g(z) \tag{42}$$

where

$$g(z) \equiv - \sqrt{\frac{\pi}{z}}\, \exp(z)\, F(z) \tag{43}$$

For $y \ll 1$ eq. (42) implies $g(z) \to 1$. Then using eqs. (43) and (31), one finds $z \cong z_{max} \cong 0.4915$. Furthermore eq. (30) implies $\lambda(z_{max}) \to 0$. One easily shows, from eqs. (40), (41), that λ is a monotonic decreasing function of z , or, what amounts to the same thing, z is a monotonic-decreasing of λ , i.e.,

$$\left.\begin{array}{l}\dfrac{d\lambda}{dz} < 0 \\[2em] \dfrac{dz}{d\lambda} < 0\end{array}\right\} \text{ for } \quad z < z_{max}$$

$$\lambda < \lambda_{max} = 2\sqrt{2\pi}$$

$$z(\lambda_{max}) \to 0 \qquad .$$

For general y , we square eq. (42) to find

$$\frac{3}{2} y^5 g + \left(\frac{3}{4} y^2 g\right)^2 - y^3 + \frac{3}{4} y^2 (1 - g) = 0 \tag{44}$$

(or for $y \neq 0$, $g \neq 0$)

$$y^3 + \frac{3}{8} g\, y^2 - \frac{2}{3g} y + \frac{(1 - g)}{g} = 0 \qquad . \tag{45}$$

This is a cubic equation, so the standard technique applies. One finds, for all positive[†] g , there are three real roots. However, there is only one genuine positive solution of eq. (45), i.e., there is one negative root, and one spurious root. To eliminate the spurious root, we

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† From eqs. (31) and (43), it can be shown that $g \geqslant 0$ in our problem.

note that we must have $y << 1$ as $g \to 1$ corresponding to our earlier solution. Also, one must have $\lim_{z \to 0} \sqrt{z}\, y(z) = 1$. It turns out that the genuine root may be written as

$$y = \frac{u}{6}\left[\frac{2\sqrt{u^3 + 6}}{u^{3/2}} \cos\left(\frac{\pi + \phi}{3}\right) - 1\right] \tag{46}$$

where

$$\phi \equiv \tan^{-1}\left[\frac{3\sqrt{3}\,(-u^2 + 2u - \tfrac{1}{2})\sqrt{4u^2 + 13u + 32}}{\sqrt{u}\left(u^4 - 45u + \frac{81}{2}\right)}\right] \tag{47}$$

$$-\frac{\pi}{2} \leqslant \phi \leqslant \frac{\pi}{2} \tag{48}$$

with $u \equiv \frac{3}{4}\, g(z)$. The substitution of this result into eq. (40), yields λ as a function of $z \equiv -e\beta\phi_m$. The numerical calculation, which gives the relation between λ and z , is shown in Fig. 3. Fig. 3 shows the limits $\lambda(z = 0) = 2\sqrt{2\pi}$ and $z(\lambda \to 0) = .4915$, as expected. For $\lambda > 2\sqrt{2\pi}$, it appears that there is no solution. For fixed λ having obtained $\phi_m (= -z/\beta e)$ from Fig. 3, one determines ϕ_o from

$$e\,\phi_o = \frac{mu_o^2}{2} - \left[\frac{\lambda \exp(\beta e \phi_m)}{2\sqrt{2\pi}\,\beta} - e\,\phi_m\right] \tag{49}$$

V. MONOTONIC SOLUTION

It should be noted that if ϕ_m is identically zero, i.e., ϕ declines monotonically to zero, eq. (21) is an identity, so that there is one less variable and one less non-trivial condition (on ϕ_o) .

Assuming that $f_e(v_\infty,\infty)$, $f_\nu(v_o,0)$ have the special forms given by eqs. (25) and (26), and $\phi_m \equiv 0$, one finds for Poisson's equation

$$\frac{d^2\phi}{dx^2} = 4\pi e\, P(\phi) \quad , \tag{50}$$

where

$$P(\phi) = n_\nu X(\varepsilon,\gamma) - \frac{n_e}{2} Y(w) \quad ; \tag{51}$$

with

$$X(\varepsilon,\gamma) \equiv \sqrt{\varepsilon^2+\gamma} - \varepsilon + \sqrt{1-\varepsilon^2}\,\sqrt{\gamma} + (1-\varepsilon^2-\gamma)\,\ln\left[\frac{1+\sqrt{\varepsilon^2+\gamma}}{\sqrt{1-\varepsilon^2}+\sqrt{\gamma}}\right]$$

$$- (1-\varepsilon^2)\,\ln\left(\frac{1+\varepsilon}{\sqrt{1-\varepsilon^2}}\right) + 2(1-\varepsilon^2-\gamma)\,\ln\left[\frac{\sqrt{1-\varepsilon^2}+\sqrt{\gamma}}{\sqrt{1-(\varepsilon^2+\gamma)}}\right] \quad , \tag{52}$$

$$Y(w) \equiv 1 - \exp(w)\left[1 - \mathrm{erf}(\sqrt{w})\right] \quad , \tag{53}$$

$$\varepsilon \equiv \sqrt{1 - \frac{2e\phi_o}{mu_o^2}} \tag{54}$$

$$\gamma \equiv \frac{2e\phi}{mu_o^2} \tag{55}$$

and

$$w \equiv e\beta\phi \tag{56}$$

Furthermore, for this case the current equation [eq. (30)] becomes

$$\varepsilon^2 = \frac{1}{\sqrt{2\pi}} \frac{n_e}{n_\nu \sqrt{m\beta u_o^2}} \tag{57}$$

For γ and ε both small, one has approximately

$$X(\varepsilon,\gamma) = X_1\left[1 + 0(\varepsilon^2 + \gamma)\right]$$

$$X_1(\varepsilon,\gamma) \equiv 2\left[\sqrt{\gamma} + \sqrt{\varepsilon^2 + \gamma} - \varepsilon\right] \qquad \gamma << 1 \quad . \tag{58}$$

On the other hand, if $\gamma >> \varepsilon^2$

$$X(\varepsilon,\gamma) = X_2\left[1 + 0(\varepsilon)\right]$$

$$X_2(\varepsilon,\gamma) \equiv 2\sqrt{\gamma} + (1 - \gamma)\ \ell n\left(\frac{1 + \sqrt{\gamma}}{1 - \sqrt{\gamma}}\right) \quad , \quad \gamma >> \varepsilon^2 \tag{59}$$

which matches up with the preceding expression for small γ .

We will now show that the precise condition on λ to have $\left(\frac{d\phi}{dx}\right)^2 \geqslant 0$ for all x , in this case is also

$$\lambda \leqslant 2\sqrt{2\pi} \quad ,$$

where λ is defined by eq. (39).

To show this, we consider $\left(\frac{d\phi}{dx}\right)^2$ as a function of $\sqrt{\phi}$. We have that

$$\frac{d}{d\sqrt{\phi}}\left(\frac{d\phi}{dx}\right)^2 = 2\sqrt{\phi}\,\frac{d}{d\phi}\left(\frac{d\phi}{dx}\right)^2 = 16\pi e\sqrt{\phi}\,P(\phi) \quad , \tag{60}$$

where eq. (50) was used.

We know that $\left.\left(\frac{d\phi}{dx}\right)^2\right|_{\phi=0} = 0$. Since $P(\phi = 0) = 0$, the first two derivatives of $\left(\frac{d\phi}{dx}\right)^2$ with respect to $\sqrt{\phi}$ will also be zero at $\sqrt{\phi} = 0$ (note: we are considering $\left(\frac{d\phi}{dx}\right)^2$ as a function of $\sqrt{\phi}$ rather than ϕ to avoid singularities in the derivatives). We will show the following:

(A) For $\lambda > 2\sqrt{2\pi}$ the first non-vanishing derivative is

$$\left.\frac{d^3}{d(\sqrt{\phi})^3}\left(\frac{d\phi}{dx}\right)^2\right|_{\sqrt{\phi}=0} < 0$$

so that for $\sqrt{\phi}$ small but finite $\left(\frac{d\phi}{dx}\right)^2$ must go negative. Thus this case is ruled out.

(B) For $\lambda = 2\sqrt{2\pi}$

$$\left.\frac{d^3}{d(\sqrt{\phi})^3}\left(\frac{d\phi}{dx}\right)^2\right|_{\sqrt{\phi}=0} = 0 \ , \quad \left.\frac{d^4}{d(\sqrt{\phi})^4}\left(\frac{d\phi}{dx}\right)^2\right|_{\sqrt{\phi}=0} > 0$$

so that $\left(\frac{d\phi}{dx}\right)^2$ starts upward from $\sqrt{\phi} = 0$; further we will show that $P(\phi) \geqslant 0$ for $0 \leqslant \phi \leqslant \phi_o$, so that $\left(\frac{d\phi}{dx}\right)^2$ is a monotone increasing function of $\sqrt{\phi}$, and is thus always $\geqslant 0$.

(C) For $\lambda < 2\sqrt{2\pi}$ we show that $P(\phi) \geqslant 0$ so again $\left(\frac{d\phi}{dx}\right)^2 \geqslant 0$.

We note that the two expressions for X given by eqs. (58) and (59) are overlapping, therefore we may assume that eq. (58) is valid for $\gamma < \varepsilon$, and eq. (59) is valid for $\gamma > \varepsilon$.

We desire to express everything in terms of a single variable. We employ eq. (57) in the expressions for X ; then wr.te

$$\gamma = \frac{2w}{\beta m u_o^2} \quad , \qquad \text{[via eqs. (55) and (56)]}$$

and introduce

$$\mu \equiv \sqrt{w} \tag{61}$$

When the dust has cleared, we find an expression of the form

$$\frac{d}{d\sqrt{\phi}}\left(\frac{d\phi}{dx}\right)^2 = \frac{32\pi n_\nu}{\alpha}\sqrt{\frac{e}{\beta}}\,\mu\left[D(\mu) - \frac{\lambda}{4\sqrt{2}}\,Y(\mu^2)\right] \equiv \text{const. } P(\mu) \quad , \tag{62}$$

where

$$D(\mu) \cong \begin{cases} \mu + \sqrt{\mu^2 + \dfrac{\lambda}{2\sqrt{2\pi}}} - \sqrt{\dfrac{\lambda}{2\sqrt{2\pi}}} \quad , & \mu < \sqrt{\dfrac{\lambda}{2\sqrt{2\pi}}} \\ \mu + \dfrac{(\alpha^2 - \mu^2)}{2\alpha}\,\ell n\left(\dfrac{\alpha + \mu}{\alpha - \mu}\right) \; , & \sqrt{\dfrac{\lambda}{2\sqrt{2\pi}}} < \mu < \sqrt{\alpha^2 - \dfrac{\lambda}{2\sqrt{2\pi}}} \end{cases} \tag{63}$$

with

$$\alpha \equiv \sqrt{\frac{\beta m u_o^2}{2}} \tag{64}$$

Where necessary, we use the fact that

$$\frac{\lambda}{\alpha^2} \propto \varepsilon^2 << 1$$

For small μ , we expand $D(\mu)$, $Y(\mu^2)$ to obtain

$$P(\mu) = \mu\left[\mu\left(1 - \frac{\lambda}{2\sqrt{2\pi}}\right) + \mu^2\left(\frac{\sqrt{2\pi}}{\lambda} + \frac{\lambda}{4\sqrt{2}}\right) + 0(\mu^3)\right] \tag{65}$$

Thus for $\lambda > 2\sqrt{2\pi}$, $P(\mu)$ is zero and decreasing in the neighborhood of $\mu = 0^+$. This implies that $\left(\frac{d\phi}{dx}\right)^2$ goes negative for μ small but finite, so we must rule out this case. For $\lambda \leqslant 2\sqrt{2\pi}$, P is increasing in the neighborhood of the origin, so $\left(\frac{d\phi}{dx}\right)^2$ is positive there.

We now ask whether $P(\mu)$ is ever negative for $\lambda \leqslant 2\sqrt{2\pi}$. As shown in Appendix I it is not. Therefore $\left(\frac{d\phi}{dx}\right)^2$ is a monotone increasing function of $\sqrt{\phi}$, and is always positive for $0 \leqslant \phi \leqslant \phi_o$ (i.e., for all $0 \leqslant x \leqslant \infty$) .

V. ENERGY CONSIDERATIONS

We have seen that for $\lambda \equiv n_e\sqrt{\beta m u_o^2}/n_\nu < 2\sqrt{2\pi}$, there are two solutions of the steady state Vlasov-Poisson system. Such a result is not especially unusual for a non-linear system. However, one would expect physically that the system under consideration should have a unique steady state potential. Thus it seems likely that one of our solutions is not a true steady state. While this question cannot be settled conclusively without solving the full time-dependent non-linear Vlasov-Poisson system, the following argument suggests that the non-monotonic solution is the stable one.

Consider the total potential energy, defined by

$$U = \int_0^\infty dx \left[\frac{1}{8\pi} \left(\frac{d\phi}{dx} \right)^2 - \rho(x)\ \phi(x) \right] - \sigma\ \phi_o \tag{66}$$

where $\rho(x)$ is the space charge density and σ the surface charge density on the plate. If we regard $\rho(x)$, σ as given and vary $\phi(x)$, we find that (66) is a minimum for any ϕ which is related to ρ,σ by Poisson's equation and the corresponding surface relation

$$\left. \frac{d\phi}{dx} \right|_{x \to o+} = - 4\pi\sigma \tag{67}$$

which follows from Poisson's equation and the fact that the field inside the metal is zero. It seems reasonable to suppose that if Poisson's equation has <u>two</u> or more solutions, the system will tend to seek the state with the lowest potential energy. In the calculation which follows, we will show that the non-monotonic solution has less potential energy than the monotonic one.

Using Poisson's equation, partial integration and Eq. (67) in (66), one finds the simplified expression

$$U = - \frac{1}{8\pi} \int_0^\infty dx \left(\frac{d\phi}{dx} \right)^2 \tag{68}$$

Using now Eq. (18) and an obvious change of integration variables, one may write (68) as

$$U = \frac{1}{4\pi\sqrt{2}} \left[\int_{\phi_m}^{\phi_o} d\phi \sqrt{-V_1(\phi)} + \int_{\phi_m}^{o} d\phi \sqrt{-V_2(\phi)} \right] \tag{69}$$

Using the special distribution functions, given by eqs. (25) and (26), in eqs. (16), and (17), with the help of eq. (20) one finds

$$V_2(\phi) = -2\pi m u_o^2 n_\nu \left\{ H(\gamma,\delta,\varepsilon) - \frac{n_e}{n_\nu(\beta m u_o^2)} K(w,z) \right\} \quad , \tag{70}$$

and

$$V_1(\phi) = -4\pi m u_o^2 n_\nu \left\{ \tfrac{1}{2} \left[H(\gamma,\delta,\varepsilon) - \frac{n_e}{n_\nu(\beta m u_o^2)} K(w,z) \right] + \left[J(\gamma,\varepsilon) - \frac{n_e}{n_\nu \beta m u_o^2} L(w,z) \right] \right\} . \tag{71}$$

where

$$\left.\begin{aligned} w &\equiv \beta e(\phi - \phi_m) \quad , \\ \gamma &\equiv \frac{2w}{\beta m u_o^2} \\ z &\equiv -\beta e\, \phi_m \\ \delta &\equiv \frac{2z}{\beta m u_o^2} \\ \varepsilon &\equiv \sqrt{1 - \frac{2e(\phi_o - \phi_m)}{m u_o^2}} \end{aligned}\right\} \tag{72}$$

$$K(w,z) \equiv w\left[1+\mathrm{erf}(\sqrt{z})\right] - \exp(w-z)\left[1+\mathrm{erf}(\sqrt{w})\right] + \exp(-z)\left[1+2\sqrt{\frac{w}{\pi}}\right] , \quad (73)$$

$$H(\gamma,\delta,\varepsilon) \equiv (\varepsilon^2+\gamma)^{3/2} - \gamma^{3/2}\sqrt{1-\varepsilon^2} + \frac{(1-\varepsilon^2-\gamma)}{2}\left(\sqrt{\varepsilon^2+\gamma} - \sqrt{1-\varepsilon^2}\sqrt{\gamma}\right) -$$

$$- \frac{(1-\varepsilon^2-\gamma^2)}{2}\,\ell n\left[\frac{1+\sqrt{\varepsilon^2+\gamma}}{\sqrt{1-\varepsilon^2}+\sqrt{\gamma}}\right] - \varepsilon^3 - \frac{\varepsilon(1-\varepsilon^2)}{2} + \frac{(1-\varepsilon^2)^2}{2}\,\ell n\left(\frac{1+\varepsilon}{\sqrt{1-\varepsilon^2}}\right)$$

$$- \gamma\left[\sqrt{\varepsilon^2+\delta} - \sqrt{1-\varepsilon^2}\sqrt{\gamma} + (1-\varepsilon^2-\gamma)\,\ell n\left(\frac{1+\sqrt{\varepsilon^2+\delta}}{\sqrt{1-\varepsilon^2}+\sqrt{\delta}}\right)\right] , \quad (74)$$

$$L(w,z) \equiv \exp(-z)\left[\exp(w)\,\mathrm{erf}\,(\sqrt{w}) - 2\sqrt{\frac{w}{\pi}}\right] , \quad (75)$$

and

$$J(\gamma,\varepsilon) \equiv \gamma^{3/2}\sqrt{1-\varepsilon^2} + \frac{(1-\varepsilon^2-\gamma)}{2}\sqrt{\gamma}\sqrt{1-\varepsilon^2} -$$

$$- \tfrac{1}{2}(1-\varepsilon^2-\gamma)^2\,\ell n\left(\frac{\sqrt{1-\varepsilon^2}+\sqrt{\gamma}}{\sqrt{1-\varepsilon^2-\gamma}}\right) \quad (76)$$

Since Eq. (70) is for the region $x \geqslant x_m$, hence we have $w \leqslant z$, $\gamma \leqslant \delta$. On the other hand, Eq. (71) is valid for the region $x \leqslant x_m$ where γ can range all the way up to $(1 - \varepsilon^2)$. Furthermore, we note from Eq. (76) $J(\gamma,\varepsilon)$ is independent of δ .

Substituting eqs. (70) and (71) into Eq. (69), and introducing γ as a new integration variable, one finds

$$U = -\frac{mu_o^2}{8e}\sqrt{\frac{mu_o^2 n_\nu}{\pi}}\left\{\int_o^\delta d\gamma\sqrt{H(\gamma,\delta,\varepsilon) - \frac{n_e}{n_\nu \beta mu_o^2} K\left(\frac{\beta mu_o^2}{2}\gamma, z\right)}\right.$$

$$\left. + \sqrt{2}\int_o^{1-\varepsilon^2} d\gamma\sqrt{J(\gamma,\varepsilon) + \frac{H(\gamma,\delta,\varepsilon)}{2} - \frac{n_e}{n_\nu \beta mu_o^2}\left[L\left(\frac{\beta mu_o^2}{2}\gamma, z\right) + \frac{K\left(\frac{\beta mu_o^2}{2}\gamma, z\right)}{2}\right]}\right\} \tag{77}$$

The same form is valid for the monotonic case provided both δ, and z are replaced by zero and the ε is modified accordingly (recall that $\varepsilon^2 \cong n_e\, e^{-z} \Big/ n_\nu \sqrt{2\pi\beta mu_o^2}$) . If we denote the potential energy of the non-monotonic and monotonic solutions as U_{NM} , and U_M respectively; then as shown in Appendix II, we have

$$U_{NM} - U_M \leqslant 0 \tag{78}$$

provided $\lambda \leqslant 2\sqrt{2\pi}$, where the equality occurs only for $\lambda = 2\sqrt{2\pi}$ (i.e. $z = 0$).

Thus it is suggested that the <u>ultimate</u> steady state potential will have the form of Fig.1, whereas the monotonic solution may be only metastable (i.e. capable of existing virtually unchanged for relatively long periods of time, but ultimately decaying to the "true" steady state).

ACKNOWLEDGEMENTS

We are indebted to D.A. Tidman for suggesting this problem and for stimulating discussions. The authors are also pleased to acknowledge many helpful conversations with R. Goldman. The work has been supported by National Aeronautics and Space Administration under Grants NGL-21-002-033 and NGL-21-002-005/7.

APPENDIX I

The proof of $P(\mu) \geqslant 0$ for $\lambda \leqslant 2\sqrt{2\pi}$

(1) First we consider the region $\mu < \sqrt{\lambda/2\sqrt{2\pi}}$. Then from eqs. (62) and (63), we have

$$P(\mu) \cong P_1(\mu) = \mu\, Q_1(\mu) \tag{A-1}$$

where

$$Q_1(\mu) \equiv \mu + \sqrt{\mu^2 + \frac{\lambda}{2\sqrt{2\pi}}} - \sqrt{\frac{\lambda}{2\sqrt{2\pi}}} - \frac{\lambda}{4\sqrt{2}}\, Y(\mu^2) \;,\quad \mu < \sqrt{\frac{\lambda}{2\sqrt{2\pi}}} \;. \tag{A-2}$$

By elementary differentiation,

$$Q_1'(\mu) = \left(1 - \frac{\lambda}{2\sqrt{2\pi}}\right) + \frac{\mu}{\sqrt{\mu^2 + \frac{\lambda}{2\sqrt{2\pi}}}} + \frac{\lambda\mu}{2\sqrt{2}} \exp\left(\mu^2\right)\left[1 - \mathrm{erf}\,(\mu)\right] \;. \tag{A-3}$$

It is obvious by inspection that

$$Q_1'(\mu) \geqslant 0$$

and the equality holds if and only if both $\lambda = 2\sqrt{2\pi}$, $\mu = 0$. Thus $Q_1(\mu)$ is monotone increasing and always greater than zero.

(2) For the second region, we have

$$P(\mu) \cong \mu\, Q_2(\mu) \;,\quad \sqrt{\frac{\lambda}{2\sqrt{2\pi}}} < \mu < \sqrt{\alpha^2 - \frac{\lambda}{2\sqrt{2\pi}}} \tag{A-4}$$

where

$$Q_2(\mu) \equiv \mu + \frac{\alpha^2 - \mu^2}{2\alpha} \ell n\left(\frac{\alpha + \mu}{\alpha - \mu}\right) - \frac{\lambda}{4\sqrt{2}} Y(\mu^2) \tag{A-5}$$

Inasmuch as $\lambda/2\sqrt{2\pi} << \alpha^2$, we have that

$$Q_2(\mu_m) \cong 2\mu_m - \frac{\sqrt{\pi}}{2} \mu_m^2 Y(\mu_m^2) \equiv T(\mu_m) \tag{A-6}$$

where

$$\mu_m \equiv \sqrt{\frac{\lambda}{2\sqrt{2\pi}}} \tag{A-7}$$

Now μ_m can be anything from zero to one. But

$$T(o) = 0 \qquad T(1) > 0 \quad ,$$

(note the last inequality is due to the fact that $Y(\mu_m^2) \leqslant 0$ for all μ_m) and

$$T'(\mu_m) = (2-\sqrt{\pi}\mu_m-\mu_m^2) + \sqrt{\pi}\mu_m(1+\mu_m^2)\exp(\mu_m^2)\left[1-\text{erf}(\mu_m)\right] > 0, 0 \leqslant \mu_m \leqslant 1 \ , \tag{A-8}$$

so $T(\mu_m)$ is monotone and therefore positive on (0,1) and $Q_2(\mu_m) > 0$.

At the other limit

$$Q_2\left(\sqrt{\alpha^2-\mu_m^2}\right) \cong \alpha\left\{1 + \frac{\mu_m^2}{\alpha 2} \ell n\left(\frac{2\alpha}{\mu_m}\right) - \mu_m^2\left[\frac{\sqrt{\pi}}{2\alpha} Y(\alpha^2)\right]\right\} \quad . \tag{A-9}$$

But $\frac{1}{\alpha} Y(\alpha^2) \leqslant \frac{2}{\sqrt{\pi}}$ so

$$Q_2\left(\sqrt{\alpha^2 - \mu_m^2}\right) > 0 \quad \text{for} \quad \mu_m^2 \equiv \frac{\lambda}{2\sqrt{2\pi}} \leqslant 1 \tag{A-10}$$

Now we return to eq. (A-4)

$$Q_2(\mu) \equiv D_2(\mu) - \frac{\lambda}{4\sqrt{2}} Y(\mu^2) \quad , \tag{A-11}$$

where

$$D_2(\mu) \equiv \mu + \frac{\alpha^2 - \mu^2}{2\alpha} \ell n\left(\frac{\alpha + \mu}{\alpha - \mu}\right)$$

From our previous result, the function $Y(\mu^2)$ is a monotonic function of μ with no inflection; whereas it can be shown that $D_2(\mu)$ has exactly one maximum (around $\mu = 0.652$) no minima and no inflection points. Since $D_2(\mu) > \lambda/4\sqrt{2}\, Y(\mu^2)$ at the endpoints, this shows that $D_2(\mu) > \lambda/4\sqrt{2}\, Y(\mu^2)$ for the whole range. Thus $Q_2(\mu) > 0$ for the entire range. Therefore

$$P(\mu) \geqslant 0 \quad \text{for} \quad \lambda \leqslant 2\sqrt{2\pi} \quad \text{Q.E.D.} \tag{A-12}$$

APPENDIX II

Proof: $U_{NM} - U_M \leqslant 0$.

We have, from Eq. (74)

$$H(\gamma,\varepsilon,\delta) = (\varepsilon^2+\gamma)^{3/2} - \gamma^{3/2}\sqrt{1-\varepsilon^2} + \frac{(1-\varepsilon^2-\gamma)}{2}\left[\sqrt{\varepsilon^2+\gamma}-\sqrt{1-\varepsilon^2}\sqrt{\gamma}\,\right] -$$

$$- \frac{(1-\varepsilon^2-\gamma)^2}{2}\,\ell n\left[\frac{1+\sqrt{\varepsilon^2+\gamma}}{\sqrt{1-\varepsilon^2}+\sqrt{\gamma}}\right] - \varepsilon^3 - \frac{\varepsilon(1-\varepsilon^2)^2}{2} + \frac{(1-\varepsilon^2)^2}{2}\,\ell n\left[\frac{1+\varepsilon}{\sqrt{1-\varepsilon^2}}\right] -$$

$$- \gamma\left[\sqrt{\varepsilon^2+\delta}-\sqrt{1-\varepsilon^2}\sqrt{\gamma} + (1-\varepsilon^2-\gamma)\;\ell n\left(\frac{1+\sqrt{\varepsilon^2+\delta}}{\sqrt{1-\varepsilon^2}+\sqrt{\delta}}\right)\right] \qquad (74)$$

We write

$$H(\gamma,\delta,\varepsilon) = M(\gamma,\varepsilon) - \gamma N(\delta,\varepsilon) \quad , \qquad \text{(B-1)}$$

and note the following limits:

(A) $\gamma \ll 1$

$$M(\gamma,\varepsilon) = \frac{4}{3}\left[(\varepsilon^2+\gamma)^{3/2}-\sqrt{1-\varepsilon^2}\;\gamma^{3/2}-\varepsilon^3\right] + O\left((\varepsilon^2+\gamma)^2\right) \quad \begin{array}{l}\gamma \ll 1\\ \varepsilon \ll 1\end{array} \qquad \text{(B-2)}$$

with the subcases

(a) $\gamma \ll \varepsilon^2$

$$M(\gamma,\varepsilon) \cong 2\varepsilon\gamma\left[1 + O\left(\frac{\sqrt{\gamma}}{\varepsilon}\right)\right] , \quad \gamma \ll \varepsilon^2 \ll 1 \qquad \text{(B-3)}$$

(b) $\varepsilon^2 \ll \gamma \ll 1$

$$M(\gamma,\varepsilon) = 2\sqrt{\gamma}\varepsilon^2\left[1 + \frac{\gamma}{3} + O\left(\frac{\varepsilon}{\sqrt{\gamma}}\right)\right] \varepsilon^2 \ll \gamma \ll 1 \qquad \text{(B-4)}$$

To take into account the possibility that γ may be near unity, we consider

(B) $\gamma >> \varepsilon^2$.

In this case a slightly different expansion may be used, and one finds

$$M(\gamma,\varepsilon) = 2\sqrt{\gamma}\varepsilon^2 \left[1 + 0\left(\frac{\varepsilon}{\sqrt{\gamma}}\right)\right] \quad , \quad \gamma >> \varepsilon^2 \tag{B-5}$$

Clearly Eq. (B-2) gives the correct leading term for all γ . Turning now to N , we have two cases.

(α) Non-monotonic: Expansions for $\delta << 1$ and $\delta >> \varepsilon^2$ match up nicely and are well approximated by

$$N(\delta,\varepsilon) = \left[2\sqrt{\varepsilon^2+\delta} - \sqrt{\delta}\right] \tag{B-6}$$

For all δ between

$$0 \leq \delta < \frac{2\ z_{max}}{\beta m u_o^2} \quad \text{with} \quad z_{max} = 0.4915$$

(β) Monotonic $\delta = 0$

$$N(0,\varepsilon) = 2\varepsilon\left[1 + 0(\varepsilon^2)\right] \tag{B-7}$$

From Eq. (76), we have

$$J(\gamma,\varepsilon) \equiv \gamma^{3/2}\sqrt{1-\varepsilon^2} + \frac{(1-\varepsilon^2-\gamma)}{2} - \frac{(1-\varepsilon^2-\gamma)^2}{2}\ \ell n\left[\frac{\sqrt{1-\varepsilon^2}+\sqrt{\gamma}}{\sqrt{1-\varepsilon^2-\gamma}}\right] \quad . \tag{76}$$

It is easy to show that

$$J(\gamma,\varepsilon) = \tfrac{1}{2}\left[\gamma^{3/2} + \sqrt{\gamma} - \frac{(1-\gamma)^2}{2}\,\ell n\left(\frac{1+\sqrt{\gamma}}{1-\sqrt{\gamma}}\right)\right]\left[1 + 0(\varepsilon^2)\right] \tag{B-8}$$

for all $\gamma < 1-\varepsilon^2$. We note that the leading term may be written as $C(\gamma)\,\gamma^{3/2}$ where $C(\gamma)$ varies only sligthly (from 4/3 to 1), for the whole range of γ . The correction is, at worst, ε^2 time the leading term. The leading term is the same for the monotonic case and the non-monotonic case, except the definition of γ .

Now the potential energy is given by Eq. (77)

$$U = -\frac{mu_o^2}{8e}\sqrt{\frac{mu_o^2 n_\nu}{\pi}}\left\{\int_o^\delta d\gamma\,\sqrt{H(\gamma,\delta,\varepsilon) - \frac{n_e}{n_\nu \beta mu_o^2}\,K\left(\frac{\beta mu_o^2}{2}\,\gamma,z\right)}\right.$$

$$\left. + \sqrt{2}\int_o^{1-\varepsilon^2} d\gamma\,\sqrt{J(\gamma,\varepsilon) + \frac{H(\gamma,\delta,\varepsilon)}{2} - \frac{n_e}{n_\nu \beta mu_o^2}\left[L\left(\frac{\beta mu_o^2}{2}\,\gamma,z\right) + \tfrac{1}{2}\,K\left(\frac{\beta mu_o^2}{2}\,\gamma,z\right)\right]}\right\} \tag{77}$$

where K , L , are given by eqs. (73) and (75), i.e.

$$K(w,z) \equiv w\left[1+\mathrm{erf}(\sqrt{z})\right] - \exp(w-z)\left[1+\mathrm{erf}(\sqrt{w})\right] + \exp(-z)\left[1+2\sqrt{\frac{w}{\pi}}\right] \tag{73}$$

and

$$L(w,z) \equiv \exp(-z)\left[\exp(w)\ \mathrm{erf}(\sqrt{w}) - 2\sqrt{\frac{w}{\pi}}\right] \tag{75}$$

with $w \equiv \beta mu_o^2/2\ \gamma$.

The expression (77) is valid for both cases. For monotonic case, both δ , and z are replaced by zero, and ε is modified accordingly (recall that $\varepsilon^2 \cong n_e \exp(-z)/n_\nu\sqrt{2\pi m\beta u_o^2})$.

Now the first integral in eq. (77) is positive for $z > 0$ (it vanishes of course, for $z = 0$) . Thus, given the minus sign outside, this term (call it $\mathcal{J}$) tends to decrease the potential energy for the non-monotonic case (it is zero for the monotonic case).

Turning to the second integral of eq. (69), we note from eqs. (B-8), (75), (B-1), (B-2), (B-6), (B-7), and (73) that J is much larger than the other terms as long as $\gamma \gg \varepsilon^2$. Accordingly, we break the second integral into two parts at some convenient point γ_o such that $\varepsilon^2 \ll \gamma_o \ll 1$. One such point is

$$\gamma_o = \varepsilon^{3/2} \tag{B-9}$$

But the maximum value of the integral on $(0,\varepsilon^{3/2})$ is $\leq \varepsilon^{9/4}$; so that

$$\int_0^{\varepsilon^{3/2}} d\gamma \sqrt{J + \frac{H}{2} - \frac{n_e}{n_\nu \beta m u_o^2}\left(L + \frac{K}{2}\right)} \leq \varepsilon^{15/4} \quad , \tag{B-10}$$

which can safely be ignored. Also it is not difficult to show from Eq. (76) that

$$J(\gamma,\varepsilon) = J_o(\gamma)\left[1 - C_1(\gamma)\varepsilon^2\right] \text{, where } \tfrac{1}{2} \leq C_1(\gamma) \leq 1 \text{ , for all } \gamma \text{ .} \tag{B-11}$$

Calling the positive constant in Eq. (77) $\mathcal{C}$, one thus has approximately

$$U \cong - C\left\{J + \sqrt{2}\int_{\varepsilon^{3/2}}^{1-\varepsilon^2} d\gamma\ \sqrt{J_o(\gamma)}\left[1 - \frac{C_1\varepsilon^2}{2} + \right.\right.$$

$$\left.\left. + \frac{1}{2J_o(\gamma)}\left(\frac{M(\gamma,\varepsilon) - \gamma N(\delta,\varepsilon)}{2} - \frac{n_e}{n_\nu m\beta u_o^2}\left(L + \frac{K}{2}\right)\right)\right]\right\} . \tag{B-12}$$

Now we make the following observations:

(1)
$$\int_{\varepsilon^{3/2}}^{(1-\varepsilon^2)} d\gamma\ \sqrt{J_o(\gamma)} = \int_o^1 d\gamma\ \sqrt{J_o(\gamma)} - \varepsilon^2 + o(\varepsilon^2) \quad , \tag{B-13}$$

(2) Using Eq. (B-4), and the fact noted above that

$$J_o(\gamma) = C_o\ \gamma^{3/2} \quad , \quad 1 \leqslant C_o \leqslant 4/3$$

$$\tfrac{1}{4}\int_{\varepsilon^{3/2}}^{1-\varepsilon^2} d\gamma\ \frac{M(\gamma,\varepsilon)}{\sqrt{J_o(\gamma)}} < \frac{2}{3}\varepsilon^2 \tag{B-14}$$

(3) From eqs. (75), (73), one readily shows that

$$L(w,z) + \frac{K(w,z)}{2} - \left[L(w,0) + \frac{K(w,0)}{2}\right] = \frac{w}{2}\ \mathrm{erf}(\sqrt{z}) + \frac{\left(1-\exp(-z)\right)}{2}\ \Psi(w) \tag{B-15}$$

where

$$\Psi(w) = \exp(w)\left[1-\mathrm{erf}(\sqrt{w})\right] - 1 - \sqrt{\frac{w}{\pi}} \leqslant 0 \tag{B-16}$$

(4) From eqs. (B-6), (B-7)

$$N(\delta,\varepsilon) - N(0,\varepsilon) = 2\left[\sqrt{\varepsilon^2+\delta}-\varepsilon-\sqrt{\delta}\right] + 2(\varepsilon-\varepsilon_o) \tag{B-17}$$

where $\varepsilon_o \equiv \varepsilon(z = \delta = 0)$. (B-18)

Clearly this is negative for all $\delta > 0$.

From these considerations it is clear that (denoting the potential energy of the non-monotonic and monotonic solutions as U_{NM} , U_N respectively)

$$U_{NM}-U_M = -\left\{J' + \alpha\varepsilon_o^2\left[1 - \exp(-z)\right] + \tfrac{1}{2}(\varepsilon_o+\sqrt{\delta}-\sqrt{\varepsilon^2+\delta})\int_o^1 d\gamma \frac{\gamma}{\sqrt{J_o(\gamma)}} - \frac{n_e}{8n_\nu}\int_o^1 d\gamma \frac{\gamma \operatorname{erf}\left(\sqrt{z}\right)}{\sqrt{J_o(\gamma)}}\right\} \tag{B-19}$$

Here J' is positive (it is essentially J plus positive contributions from the term involving $\Psi(w)$ and α is positive and $O(1)$.

Now it is very simple to show that the curly bracket in Eq. (B-19) is positive, provided $\lambda \leqslant 2\sqrt{2\pi}$. Even for $z \ll 1$ the last (negative) term has an order of magnitude $[(\lambda/2\sqrt{2\pi})\int_o^1 d\gamma\ \gamma/\sqrt{J_o(\gamma)}\ ]$. Thus

$$U_{NM} - U_M \leqslant 0 \quad , \quad \frac{\lambda}{2\sqrt{2\pi}} \leqslant 1$$

where the equality occurs only for $z = 0$.

REFERENCES

1. Singer, S.F. "Scientific Uses of Earth Satellites" ed. J.A. Van Allen, p.304, The University of Michigan Press, 1956.

2. Whipple, E.C. Jr. "The Equilibrium Electric Potential of a Body in the Upper Atmosphere and in Interplanetary Space", NASA x-615-65-296, 1965.

3. Satellites measurements suggest that both types of potential distributions occur in practice, according to Dr. J. Heppner of Goddard Space Flight Center, NASA (private communication).

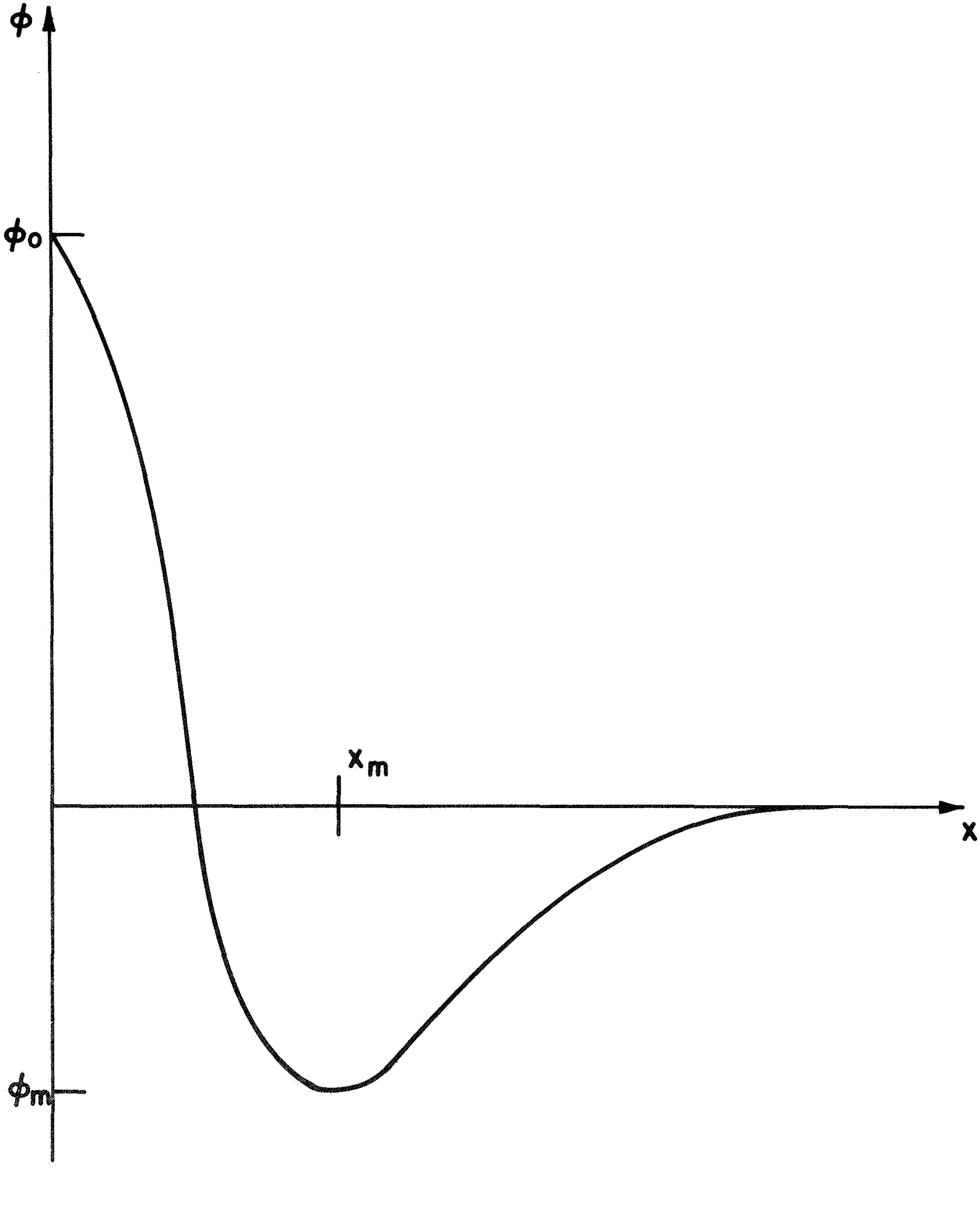

Fig. I. NON-MONOTONIC POTENTIAL DISTRIBUTION.

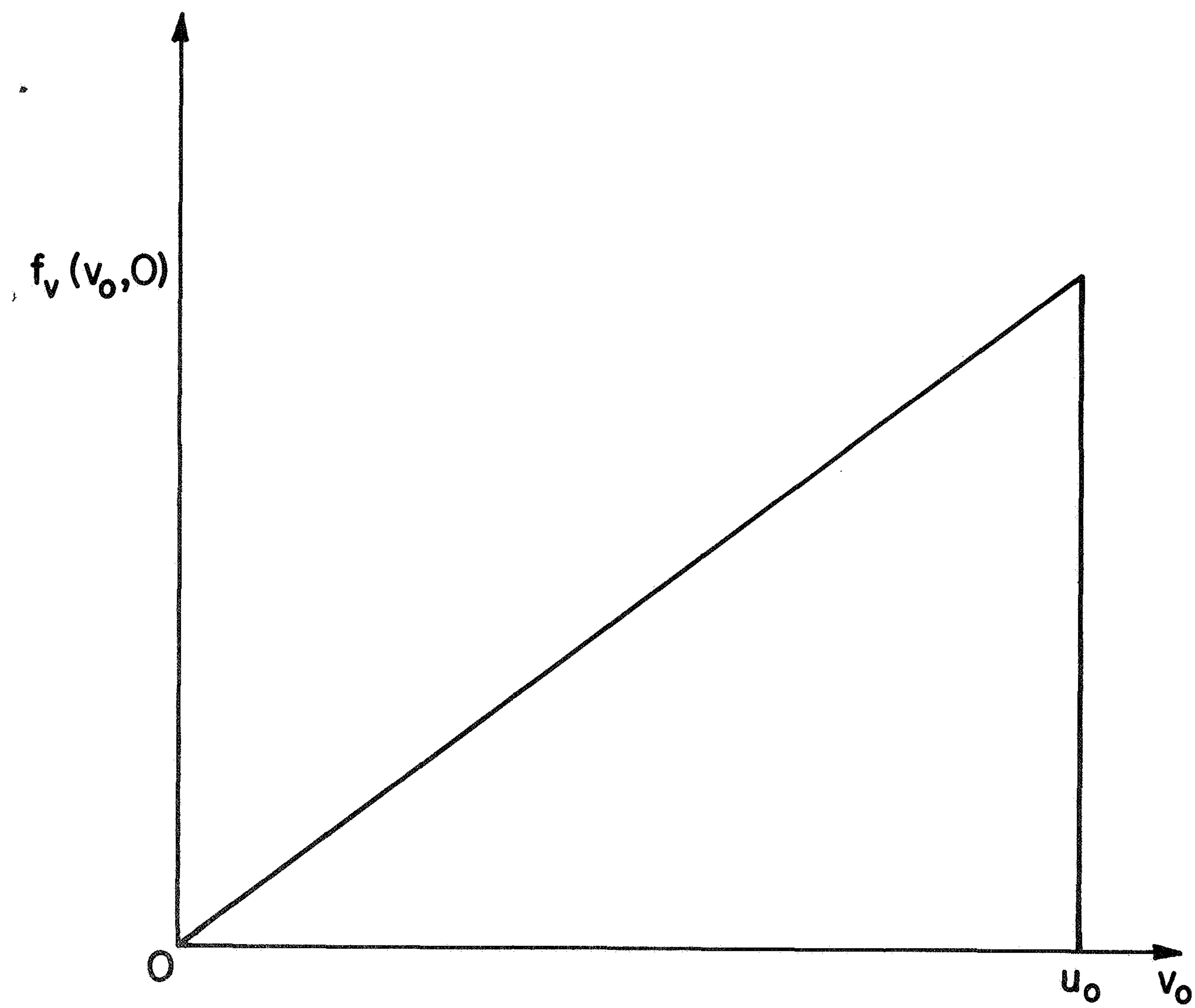

Fig. 2. ASSUMED DISTRIBUTION OF THE EMITTED PHOTO-ELECTRONS.

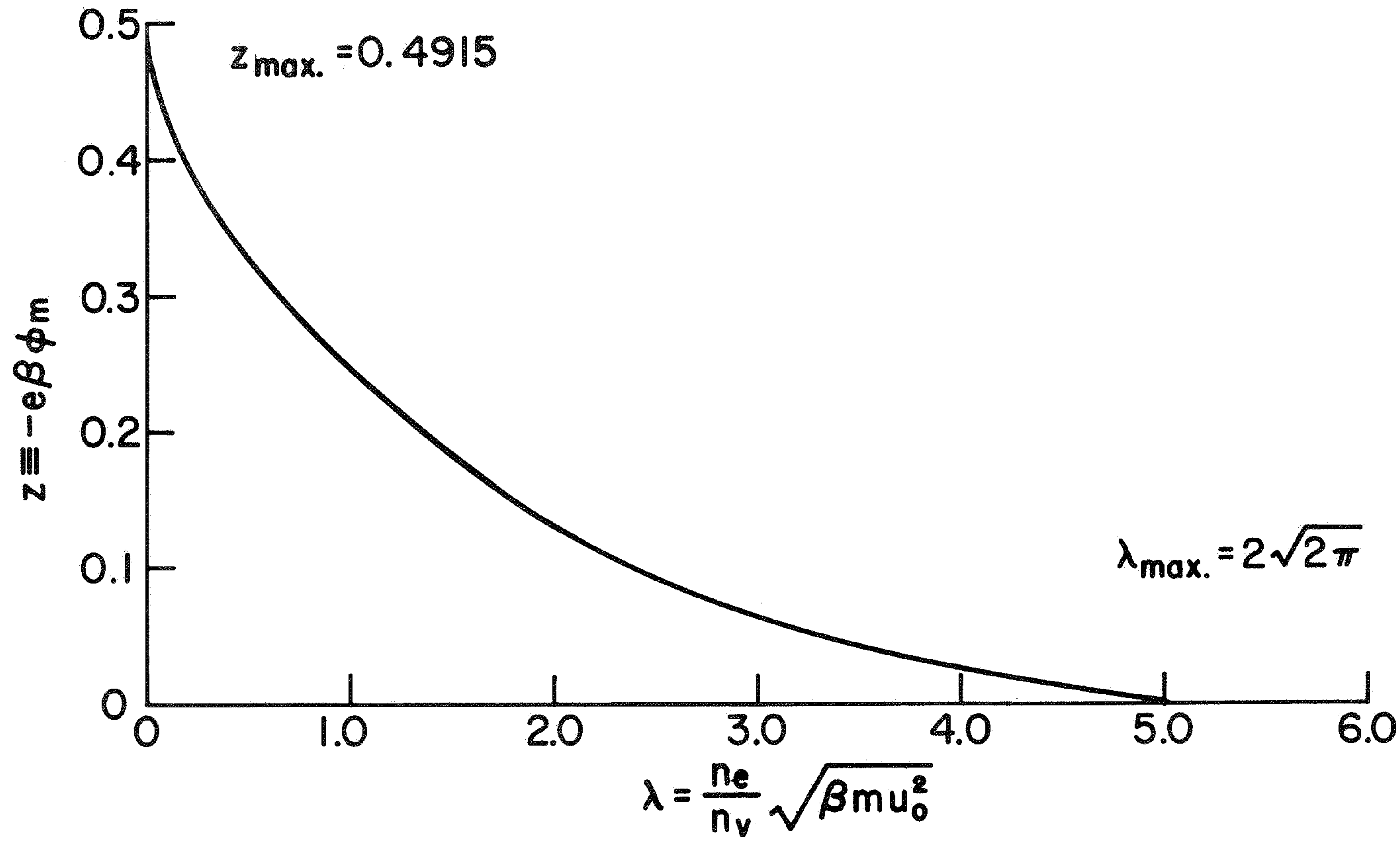

Fig. 3. MAGNITUDE OF POTENTIAL MINIMUM vs. PARAMETER λ OF THE SYSTEM.